Harmonic oscillator action

This is the Lagrangian for a harmonic oscillator.

$$L = \frac{m(\dot{x}^2 - \omega^2 x^2)}{2}$$

Show that

$$S = \int_0^T L\,dt = \frac{m\omega}{2\sin\omega T}\left((x_b^2 + x_a^2)\cos\omega T - 2x_b x_a\right), \quad T = t_b - t_a$$

The first step is to derive $x(t)$ and $\dot{x}(t)$ from L and the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$$

From L we have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x}, \quad \frac{\partial L}{\partial x} = -m\omega^2 x$$

and by Euler-Lagrange

$$\ddot{x}(t) = -\omega^2 x \tag{1}$$

The well-known solution to (1) is

$$x(t) = A\sin(\omega t) + B\cos(\omega t)$$

We have the following boundary conditions.

$$x(0) = x_a, \quad x(T) = x_b \tag{2}$$

Solve for B.

$$B = x(0) = x_a$$

For $x(T)$ we have

$$x(T) = A\sin(\omega T) + B\cos(\omega T)$$

Solve for A.

$$A = \frac{x(T) - B\cos(\omega T)}{\sin(\omega T)} = \frac{x_b - x_a\cos(\omega T)}{\sin(\omega T)}$$

Hence the equation of motion is

$$\begin{aligned} x(t) &= A\sin(\omega t) + B\cos(\omega t) \\ &= \frac{x_b - x_a\cos(\omega T)}{\sin(\omega T)}\sin(\omega t) + x_a\cos(\omega t) \end{aligned} \tag{3}$$

Take the time derivative of $x(t)$ to obtain $\dot{x}(t)$.

$$\dot{x}(t) = \frac{d}{dt}x(t) = \frac{x_b - x_a\cos(\omega T)}{\sin(\omega T)}\omega\cos(\omega t) - x_a\omega\sin(\omega t) \tag{4}$$

The action is

$$\begin{aligned} S &= \int_0^T L\,dt \\ &= \frac{m}{2}\int_0^T (\dot{x}^2 - \omega^2 x^2)\,dt \\ &= \frac{m}{2}\left(\int_0^T \dot{x}^2\,dt - \int_0^T \omega^2 x^2\,dt\right) \end{aligned}$$

Use integration by parts to solve the first integral. Let

$$u = \dot{x}, \quad v' = \dot{x}$$

so that

$$u' = \ddot{x}, \quad v = x$$

The integral transforms as

$$\begin{aligned} \int_0^T \dot{x}^2\,dt &= \int_0^T uv'\,dt \\ &= [uv]_0^T - \int_0^T u'v\,dt \\ &= \dot{x}(T)x(T) - \dot{x}(0)x(0) - \int_0^T \ddot{x}x\,dt \end{aligned}$$

Hence

$$S = \frac{m}{2}\left(\dot{x}(T)x(T) - \dot{x}(0)x(0) - \int_0^T \ddot{x}x\,dt - \int_0^T \omega^2 x^2\,dt\right)$$

The remaining integrals cancel by $\ddot{x} = -\omega^2 x$ from equation (1) leaving

$$S = \frac{m}{2}\big(\dot{x}(T)x(T) - \dot{x}(0)x(0)\big) \tag{5}$$

From the boundary conditions (2)

$$S = \frac{m}{2}\big(\dot{x}(T)x_b - \dot{x}(0)x_a\big)$$

From equation (3)

$$\dot{x}(0) = \frac{\omega\big(x_b - x_a\cos(\omega T)\big)}{\sin(\omega T)} \tag{6}$$

and

$$\dot{x}(T) = \frac{\omega\big(x_b\cos(\omega T) - x_a\big)}{\sin(\omega T)} \tag{7}$$

Hence

$$\begin{aligned} S &= \frac{m\omega}{2\sin(\omega T)}\Big[\big(x_b\cos(\omega T) - x_a\big)x_b - \big(x_b - x_a\cos(\omega T)\big)x_a\Big] \\ &= \frac{m\omega}{2\sin(\omega T)}\big((x_b^2 + x_a^2)\cos(\omega T) - 2x_b x_a\big) \end{aligned} \tag{8}$$