Harmonic oscillator 4

Let X_{kj} and P_{kj} be the matrix elements

$$X_{kj} = \int_{-\infty}^{\infty} \psi_k^* x \psi_j \, dx$$

$$P_{kj} = \int_{-\infty}^{\infty} \psi_k^* p \psi_j \, dx$$

From the previous section we have for $x\psi_j$ and $p\psi_j$

$$x\psi_j = \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{j+1}\psi_{j+1} + \sqrt{j}\psi_{j-1} \right)$$

$$p\psi_j = i\sqrt{\frac{m\hbar\omega}{2}} \left(\sqrt{j+1}\psi_{j+1} - \sqrt{j}\psi_{j-1} \right)$$

Hence by orthogonality of ψ_k and ψ_j

$$X_{kj} = \int_{-\infty}^{\infty} \psi_k^* x \psi_j \, dx = \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{j+1}\delta_{k,j+1} + \sqrt{j}\delta_{k,j-1} \right)$$

$$P_{kj} = \int_{-\infty}^{\infty} \psi_k^* p \psi_j \, dx = i\sqrt{\frac{m\hbar\omega}{2}} \left(\sqrt{j+1}\delta_{k,j+1} - \sqrt{j}\delta_{k,j-1} \right)$$

For 4×4 matrices

$$X = \sqrt{\frac{\hbar}{2m\omega}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$P = i\sqrt{\frac{m\hbar\omega}{2}} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -\sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

For the commutator $[X, P]$ we obtain

$$XP - PX = \begin{pmatrix} i\hbar & 0 & 0 & 0 \\ 0 & i\hbar & 0 & 0 \\ 0 & 0 & i\hbar & 0 \\ 0 & 0 & 0 & \boxed{-3i\hbar} \end{pmatrix} = i\hbar$$

The boxed value is incorrect due to finite matrix size.

Let H be the harmonic oscillator Hamiltonian.

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2$$

The result is

$$H = \begin{pmatrix} \frac{1}{2}\hbar\omega & 0 & 0 & 0 \\ 0 & \frac{3}{2}\hbar\omega & 0 & 0 \\ 0 & 0 & \frac{5}{2}\hbar\omega & 0 \\ 0 & 0 & 0 & \boxed{\frac{3}{2}\hbar\omega} \end{pmatrix} = \left(n + \tfrac{1}{2}\right)\hbar\omega$$

The boxed value is incorrect due to finite matrix size.

Let Ψ be the normalized state

$$\Psi = c_0\psi_0 + c_1\psi_1 + c_2\psi_2 + \cdots$$

Let us compute the expected value of x for state Ψ.

$$\langle x \rangle = \int_{-\infty}^{\infty} \Psi^* x \Psi \, dx$$

Expand the integrand.

$$\langle x \rangle = \int_{-\infty}^{\infty} \left(\sum_k c_k^* \psi_k^* \right) x \left(\sum_j c_j \psi_j \right) dx$$

Hence

$$\begin{aligned} \langle x \rangle &= \sum_k \sum_j c_k^* c_j \int_{-\infty}^{\infty} \psi_k^* x \psi_j \, dx \\ &= \sum_k \sum_j c_k^* c_j X_{kj} \end{aligned}$$

For example, let C be the coefficient vector

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

such that

$$C^\dagger C = 1$$

Then

$$\langle x \rangle = C^\dagger X C = \sqrt{\frac{\hbar}{2m\omega}} \tag{1}$$

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