Harmonic oscillator 2

From the previous section we have

$$\psi_n(x) = \frac{(-1)^n}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left(\frac{\hbar}{m\omega}\right)^{n/2} \exp\left(\frac{m\omega x^2}{2\hbar}\right) \frac{d^n}{dx^n} \exp\left(-\frac{m\omega x^2}{\hbar}\right)$$

Take the derivative of $\psi_n(x)$ to obtain

$$\frac{d}{dx}\psi_n(x) = \frac{m\omega x}{\hbar}\psi_n(x) + \frac{(-1)^n}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left(\frac{\hbar}{m\omega}\right)^{n/2} \exp\left(\frac{m\omega x^2}{2\hbar}\right) \frac{d^{n+1}}{dx^{n+1}} \exp\left(-\frac{m\omega x^2}{\hbar}\right)$$

Substitute $\psi_{n+1}(x)$ for the right-hand side.

$$\frac{d}{dx}\psi_n(x) = \frac{m\omega x}{\hbar}\psi_n(x) + \frac{\sqrt{2\,(n+1)}}{(-1)} \left(\frac{\hbar}{m\omega}\right)^{-1/2} \psi_{n+1}(x)$$

Hence

$$\left(\frac{\hbar}{2m\omega}\right)^{1/2} \left[\frac{m\omega x}{\hbar}\psi_n(x) - \frac{d}{dx}\psi_n(x)\right] = \sqrt{n+1}\,\psi_{n+1}(x)$$

By inspection of this result we define step-up operator $\hat{a}^\dagger$ as

$$\hat{a}^\dagger = \left(\frac{\hbar}{2m\omega}\right)^{1/2} \left[\frac{m\omega x}{\hbar} - \frac{d}{dx}\right]$$

Hence

$$\hat{a}^\dagger \psi_n(x) = \sqrt{n+1}\,\psi_{n+1}(x)$$

Let $\hat{a}$ be the step-down operator

$$\hat{a} = \left(\frac{\hbar}{2m\omega}\right)^{1/2} \left[\frac{m\omega x}{\hbar} + \frac{d}{dx}\right]$$

Hence (how to prove?)

$$\hat{a}\psi_n(x) = \sqrt{n}\,\psi_{n-1}(x)$$

Let $\hat{N}$ be the number operator

$$\hat{N} = \hat{a}^\dagger \hat{a}$$

Hence

$$\hat{N}\psi_n = \hat{a}^\dagger \hat{a}\psi_n = \hat{a}^\dagger \left(\sqrt{n}\psi_{n-1}\right) = n\psi_n$$

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