Free particle propagator 2

Given that at time $t = 0$

$$\psi(x) = C \exp\left(\frac{ipx}{\hbar}\right)$$

show that

$$\psi(x,t) = C \exp\left(\frac{ipx}{\hbar} - \frac{ip^2 t}{2m\hbar}\right)$$

This is the free particle propagator.

$$K(x_b, t_b, x_a, t_a) = \left(\frac{m}{2\pi i\hbar(t_b - t_a)}\right)^{\frac{1}{2}} \exp\left(\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)}\right)$$

By definition of a propagator

$$\begin{aligned}
\psi(x_b, t_b) &= \int_{-\infty}^{\infty} K(x_b, t_b, x_a, 0)\psi(x_a)\, dx_a \\
&= C\left(\frac{m}{2\pi i\hbar t_b}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \exp\left(\frac{im(x_b - x_a)^2}{2\hbar t_b} + \frac{ipx_a}{\hbar}\right) dx_a
\end{aligned}$$

Let

$$a = -\frac{im}{2\hbar t_b}, \quad b = -\frac{imx_b}{\hbar t_b} + \frac{ip}{\hbar}, \quad c = \frac{imx_b^2}{2\hbar t_b}$$

so that

$$\psi(x_b, t_b) = C\left(\frac{m}{2\pi i\hbar t_b}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \exp(-ax_a^2 + bx_a + c)\, dx_a \tag{1}$$

Solve the integral.

$$\begin{aligned}
\psi(x_b, t_b) &= C\left(\frac{m}{2\pi i\hbar t_b}\right)^{\frac{1}{2}} \left(\frac{\pi}{a}\right)^{\frac{1}{2}} \exp\left(\frac{b^2}{4a} + c\right) \\
&= C \exp\left(\frac{ipx_b}{\hbar} - \frac{ip^2 t_b}{2m\hbar}\right) \qquad (2)
\end{aligned}$$

Substitute x for x_b and t for t_b.

$$\psi(x,t) = C \exp\left(\frac{ipx}{\hbar} - \frac{ip^2 t}{2m\hbar}\right)$$

Noting that $E = p^2/2m$ we can also write

$$\psi(x,t) = C \exp\left(\frac{i(px - Et)}{\hbar}\right)$$