

Free particle propagator 1

The propagator $K(x_b, t_b, x_a, t_a)$ is the amplitude for a particle to go from x_a at time t_a to x_b at time t_b .

Show that for a free particle

$$K(x_b, t_b, x_a, t_a) = \left(\frac{m}{2\pi i \hbar (t_b - t_a)} \right)^{\frac{1}{2}} \exp \left(\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)} \right)$$

Start with the amplitude

$$K(x_b, t_b, x_a, t_a) = \langle x_b | \exp \left(-\frac{i\hat{H}(t_b - t_a)}{\hbar} \right) | x_a \rangle \quad (1)$$

where \hat{H} is the free particle Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} \quad (2)$$

Substitute (2) into (1) to obtain

$$K(x_b, t_b, x_a, t_a) = \langle x_b | \exp \left(-\frac{i\hat{p}^2(t_b - t_a)}{2m\hbar} \right) | x_a \rangle$$

By the identity

$$\int_{-\infty}^{\infty} |p\rangle \langle p| dp = 1$$

we can write

$$K(x_b, t_b, x_a, t_a) = \int_{-\infty}^{\infty} \langle x_b | \exp \left(-\frac{i\hat{p}^2(t_b - t_a)}{2m\hbar} \right) | p \rangle \langle p | x_a \rangle dp$$

By the identity $\hat{p}|p\rangle = p|p\rangle$ replace operator \hat{p} with its eigenvalue p .

$$K(x_b, t_b, x_a, t_a) = \int \langle x_b | \exp \left(-\frac{ip^2(t_b - t_a)}{2m\hbar} \right) | p \rangle \langle p | x_a \rangle dp$$

Rearrange as

$$K(x_b, t_b, x_a, t_a) = \int_{-\infty}^{\infty} \exp \left(-\frac{ip^2(t_b - t_a)}{2m\hbar} \right) \langle x_b | p \rangle \langle p | x_a \rangle dp$$

For a free particle

$$\langle x | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp \left(\frac{ipx}{\hbar} \right)$$

Hence

$$K(x_b, t_b, x_a, t_a) = \int_{-\infty}^{\infty} \exp \left(-\frac{ip^2(t_b - t_a)}{2m\hbar} \right) \frac{1}{\sqrt{2\pi\hbar}} \exp \left(\frac{ipx_b}{\hbar} \right) \frac{1}{\sqrt{2\pi\hbar}} \exp \left(-\frac{ipx_a}{\hbar} \right) dp$$

Simplify as

$$K(x_b, t_b, x_a, t_a) = \frac{1}{2m\hbar} \int_{-\infty}^{\infty} \exp\left(-\frac{ip^2(t_b - t_a)}{2m\hbar} + \frac{ip(x_b - x_a)}{\hbar}\right) dp$$

Let

$$a = \frac{i(t_b - t_a)}{2m\hbar}, \quad b = \frac{i(x_b - x_a)}{\hbar}$$

so that

$$K(x_b, t_b, x_a, t_a) = \frac{1}{2m\hbar} \int_{-\infty}^{\infty} \exp(-ap^2 + bp) dp$$

Solve the integral.

$$\begin{aligned} K(x_b, t_b, x_a, t_a) &= \frac{1}{2m\hbar} \left(\frac{\pi}{a}\right)^{\frac{1}{2}} \exp\left(\frac{b^2}{4a}\right) \\ &= \frac{1}{2\pi\hbar} \left(\frac{2\pi m\hbar}{i(t_b - t_a)}\right)^{\frac{1}{2}} \exp\left(-\frac{m(x_b - x_a)^2}{2i\hbar(t_b - t_a)}\right) \end{aligned}$$

Rewrite as

$$K(x_b, t_b, x_a, t_a) = \left(\frac{m}{2\pi i\hbar(t_b - t_a)}\right)^{\frac{1}{2}} \exp\left(\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)}\right) \quad (3)$$