Ehrenfest theorem

The Ehrenfest theorem tells us expectation values obey the laws of classical mechanics. For example, by the classical momentum formula $p = mv$ we have

$$\langle p\rangle = m\frac{d}{dt}\langle x\rangle \tag{1}$$

To verify (1), start with momentum expectation value $\langle p\rangle$.

$$\langle p\rangle = \int \psi^* p\psi\,dx$$

By the substitution

$$p = \frac{im}{\hbar}[H,x]$$

we have

$$\langle p\rangle = \frac{im}{\hbar}\int \psi^*[H,x]\psi\,dx$$

Expand the commutator.

$$\langle p\rangle = \frac{im}{\hbar}\int \psi^* Hx\psi\,dx - \frac{im}{\hbar}\int \psi^* xH\psi\,dx$$

By the Schrodinger equation we have the substitutions

$$H\psi = i\hbar\frac{d}{dt}\psi$$

and

$$\psi^* H = (H\psi)^* = -i\hbar\frac{d}{dt}\psi^*$$

Hence

$$\langle p\rangle = \frac{im}{\hbar}\int -i\hbar\frac{d}{dt}\psi^* x\psi\,dx - \frac{im}{\hbar}\int \psi^* x i\hbar\frac{d}{dt}\psi\,dx$$

Cancel $\hbar$.

$$\langle p\rangle = m\int \frac{d}{dt}\psi^* x\psi\,dx + m\int \psi^* x\frac{d}{dt}\psi\,dx$$

By the product rule for differentiation

$$\langle p\rangle = m\frac{d}{dt}\int \psi^* x\psi\,dx$$

Hence

$$\langle p\rangle = m\frac{d}{dt}\langle x\rangle$$

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