Dirac equation

This is Dirac's equation.

$$i\hbar\left(\frac{\gamma^0}{c}\frac{\partial}{\partial t}+\gamma^1\frac{\partial}{\partial x}+\gamma^2\frac{\partial}{\partial y}+\gamma^3\frac{\partial}{\partial z}\right)\psi=mc\psi$$

Gamma matrices for the "Dirac representation" are

$$\gamma^0=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\qquad\gamma^1=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}$$

$$\gamma^2=\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix}\qquad\gamma^3=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix}$$

Let ϕ be the field

$$\phi=p_xx+p_yy+p_zz-Et$$

where

$$E=\sqrt{p_x^2c^2+p_y^2c^2+p_z^2c^2+m^2c^4}$$

The four positive wave solutions to the Dirac equation are

$$\psi_1=\begin{pmatrix}E/c+mc\\0\\p_z\\p_x+ip_y\end{pmatrix}\exp\left(\frac{i\phi}{\hbar}\right)\qquad\psi_2=\begin{pmatrix}0\\E/c+mc\\p_x-ip_y\\-p_z\end{pmatrix}\exp\left(\frac{i\phi}{\hbar}\right)$$

$$\psi_3=\begin{pmatrix}p_z\\p_x+ip_y\\E/c-mc\\0\end{pmatrix}\exp\left(\frac{i\phi}{\hbar}\right)\qquad\psi_4=\begin{pmatrix}p_x-ip_y\\-p_z\\0\\E/c-mc\end{pmatrix}\exp\left(\frac{i\phi}{\hbar}\right)$$

The four negative wave solutions are

$$\psi_5=\begin{pmatrix}E/c-mc\\0\\p_z\\p_x+ip_y\end{pmatrix}\exp\left(-\frac{i\phi}{\hbar}\right)\qquad\psi_6=\begin{pmatrix}0\\E/c-mc\\p_x-ip_y\\-p_z\end{pmatrix}\exp\left(-\frac{i\phi}{\hbar}\right)$$

$$\psi_7=\begin{pmatrix}p_z\\p_x+ip_y\\E/c+mc\\0\end{pmatrix}\exp\left(-\frac{i\phi}{\hbar}\right)\qquad\psi_8=\begin{pmatrix}p_x-ip_y\\-p_z\\0\\E/c+mc\end{pmatrix}\exp\left(-\frac{i\phi}{\hbar}\right)$$

Negative wave solutions flip the sign of the mc term.

The following solutions are used for fermion fields.

$$
\begin{array}{ll}
\psi_1 & \text{fermion, spin up} \\
\psi_2 & \text{fermion, spin down} \\
\\
\psi_7 & \text{anti-fermion, spin up} \\
\psi_8 & \text{anti-fermion, spin down}
\end{array}
$$

Here is a check of physical units. The momenta p_x, p_y, and p_z have units of

$$\frac{\text{kilogram meter}}{\text{second}}$$

Hence

$$p_x x \propto \frac{\text{kilogram meter}^2}{\text{second}}$$

For the time-dependent term

$$Et \propto \frac{\text{kilogram meter}^2}{\text{second}^2} \times \text{second} = \frac{\text{kilogram meter}^2}{\text{second}}$$

We have for the reduced Planck constant

$$\hbar \propto \frac{\text{kilogram meter}^2}{\text{second}}$$

Hence $\phi/\hbar$ is dimensionless as required by the exponential function.

$$\frac{p_x x - Et}{\hbar} \propto \frac{\text{kilogram meter}^2}{\text{second}} \times \frac{\text{second}}{\text{kilogram meter}^2} = 1$$

The derivatives introduce inverse units.

$$\frac{\partial \psi}{\partial t} \propto \frac{1}{\text{second}} \qquad \frac{\partial \psi}{\partial x} \propto \frac{1}{\text{meter}}$$

Hence

$$\frac{\hbar}{c}\frac{\partial \psi}{\partial t} \propto \frac{\text{kilogram meter}}{\text{second}}$$

and

$$\hbar \frac{\partial \psi}{\partial x} \propto \frac{\text{kilogram meter}}{\text{second}}$$

The resulting units match the right-hand side of the Dirac equation.

$$mc \propto \frac{\text{kilogram meter}}{\text{second}}$$