

Dirac equation 4

From the previous section we have wavefunctions

$$\begin{aligned}
 \psi_1 &= \sqrt{E + mc^2} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z c}{E + mc^2} \\ \frac{(p_x + ip_y)c}{E + mc^2} \end{pmatrix} e^{-i\xi/\hbar} & \psi_2 &= \sqrt{E + mc^2} \begin{pmatrix} 0 \\ 1 \\ \frac{(p_x - ip_y)c}{E + mc^2} \\ \frac{-p_z c}{E + mc^2} \end{pmatrix} e^{-i\xi/\hbar} \\
 &\text{wavefunction for fermion spin up} & & \text{wavefunction for fermion spin down} \\
 \psi_3 &= \sqrt{E + mc^2} \begin{pmatrix} \frac{p_z c}{E + mc^2} \\ \frac{(p_x + ip_y)c}{E + mc^2} \\ 1 \\ 0 \end{pmatrix} e^{i\xi/\hbar} & \psi_4 &= \sqrt{E + mc^2} \begin{pmatrix} \frac{(p_x - ip_y)c}{E + mc^2} \\ \frac{-p_z c}{E + mc^2} \\ 0 \\ 1 \end{pmatrix} e^{i\xi/\hbar} \\
 &\text{wavefunction for antifermion spin up} & & \text{wavefunction for antifermion spin down}
 \end{aligned}$$

where

$$\xi = p_\mu x^\mu = Et - p_x x - p_y y - p_z z$$

and

$$E = \sqrt{p_x^2 c^2 + p_y^2 c^2 + p_z^2 c^2 + m^2 c^4}$$

Spinors u and v are equivalent to ψ with the exponentials canceled out.

$$\begin{aligned}
 u_1 &= \psi_1 e^{i\xi/\hbar} = \sqrt{E + mc^2} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z c}{E + mc^2} \\ \frac{(p_x + ip_y)c}{E + mc^2} \end{pmatrix} & u_2 &= \psi_2 e^{i\xi/\hbar} = \sqrt{E + mc^2} \begin{pmatrix} 0 \\ 1 \\ \frac{(p_x - ip_y)c}{E + mc^2} \\ \frac{-p_z c}{E + mc^2} \end{pmatrix} \\
 &\text{spinor for fermion spin up} & & \text{spinor for fermion spin down} \\
 v_1 &= \psi_3 e^{-i\xi/\hbar} = \sqrt{E + mc^2} \begin{pmatrix} \frac{p_z c}{E + mc^2} \\ \frac{(p_x + ip_y)c}{E + mc^2} \\ 1 \\ 0 \end{pmatrix} & v_2 &= \psi_4 e^{-i\xi/\hbar} = \sqrt{E + mc^2} \begin{pmatrix} \frac{(p_x - ip_y)c}{E + mc^2} \\ \frac{-p_z c}{E + mc^2} \\ 0 \\ 1 \end{pmatrix} \\
 &\text{spinor for antifermion spin up} & & \text{spinor for antifermion spin down}
 \end{aligned}$$

Spinors are solutions to the momentum-space Dirac equations

$$\not{p}u = mcu$$

$$\not{p}v = -mcv$$

where

$$\not{p} = p^\mu g_{\mu\nu} \gamma^\nu$$

and

$$p^\mu = \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

Spinors have the “completeness property”

$$\begin{aligned}u_1 \bar{u}_1 + u_2 \bar{u}_2 &= \not{p}c + mc^2 \\v_1 \bar{v}_1 + v_2 \bar{v}_2 &= \not{p}c - mc^2\end{aligned}$$

Adjoint spinors are formed as

$$\bar{u} = u^\dagger \gamma^0 \quad \bar{v} = v^\dagger \gamma^0$$

The products $u\bar{u}$ and $v\bar{v}$ are outer products that form 4×4 matrices.

Eigenmath script