Dirac equation 2

Let

$$\psi_1 = \sqrt{E + mc^2} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z c}{E+mc^2} \\ \frac{(p_x + ip_y)c}{E+mc^2} \end{pmatrix} e^{-i\xi/\hbar}$$

wavefunction for fermion spin up

$$\psi_2 = \sqrt{E + mc^2} \begin{pmatrix} 0 \\ 1 \\ \frac{(p_x - ip_y)c}{E+mc^2} \\ \frac{-p_z c}{E+mc^2} \end{pmatrix} e^{-i\xi/\hbar}$$

wavefunction for fermion spin down

$$\psi_3 = \sqrt{E + mc^2} \begin{pmatrix} \frac{p_z c}{E+mc^2} \\ \frac{(p_x + ip_y)c}{E+mc^2} \\ 1 \\ 0 \end{pmatrix} e^{i\xi/\hbar}$$

wavefunction for antifermion spin up

$$\psi_4 = \sqrt{E + mc^2} \begin{pmatrix} \frac{(p_x - ip_y)c}{E+mc^2} \\ \frac{-p_z c}{E+mc^2} \\ 0 \\ 1 \end{pmatrix} e^{i\xi/\hbar}$$

wavefunction for antifermion spin down

where

$$\xi = p_\mu x^\mu = Et - p_x x - p_y y - p_z z$$

and

$$E = \sqrt{p_x^2 c^2 + p_y^2 c^2 + p_z^2 c^2 + m^2 c^4}$$

Verify that ψ_1 et cetera are solutions to the Dirac equation

$$i\hbar \left(\frac{1}{c}\gamma^0 \frac{\partial}{\partial t} + \gamma^1 \frac{\partial}{\partial x} + \gamma^2 \frac{\partial}{\partial y} + \gamma^3 \frac{\partial}{\partial z} \right) \psi = mc\psi$$

and that the solutions are normalized as

$$|\psi|^2 = 2E$$

Gamma matrices for the "Dirac representation" are

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Let us check physical dimensions.

For the Et term in ξ we have

$$Et \propto \frac{\text{kilogram meter}^2}{\text{second}^2} \times \text{second} = \frac{\text{kilogram meter}^2}{\text{second}}$$

The momenta p_x, p_y, and p_z have units of

$$\frac{\text{kilogram meter}}{\text{second}}$$

Hence

$$-p_x x - p_y y - p_z z \propto \frac{\text{kilogram meter}}{\text{second}} \times \text{meter} = \frac{\text{kilogram meter}^2}{\text{second}}$$

We have for the reduced Planck constant

$$\hbar \propto \frac{\text{kilogram meter}^2}{\text{second}}$$

Hence $\xi/\hbar$ is dimensionless as required by the exponential function:

$$\frac{Et - p_x x - p_y y - p_z z}{\hbar} \propto \frac{\text{kilogram meter}^2}{\text{second}} \times \frac{\text{second}}{\text{kilogram meter}^2} = 1$$

From the normalization property we have

$$\psi \propto \text{joule}^{1/2}$$

The derivatives introduce inverse units.

For the time derivative

$$\frac{\partial \psi}{\partial t} \propto \frac{1}{\text{second}} \times \text{joule}^{1/2}$$

For the spatial derivatives

$$\frac{\partial \psi}{\partial x} \propto \frac{1}{\text{meter}} \times \text{joule}^{1/2}$$

Hence

$$\frac{\hbar}{c}\frac{\partial \psi}{\partial t} \propto \frac{\text{kilogram meter}}{\text{second}} \times \text{joule}^{1/2}$$

and

$$\hbar \frac{\partial \psi}{\partial x} \propto \frac{\text{kilogram meter}}{\text{second}} \times \text{joule}^{1/2}$$

The resulting units match the right-hand side of the Dirac equation.

$$mc\psi \propto \frac{\text{kilogram meter}}{\text{second}} \times \text{joule}^{1/2}$$

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