Constant force action

This is the Lagrangian for a particle in a region of constant force F.

$$L = \frac{m\dot{x}^2}{2} + Fx$$

Show that

$$S = \int_0^T L\,dt = \frac{m(x_b - x_a)^2}{2T} + \frac{FT(x_b + x_a)}{2} - \frac{F^2T^3}{24m}, \quad T = t_b - t_a$$

The first step is to derive $x(t)$ and $\dot{x}(t)$ from L and the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$$

From L we have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x}, \quad \frac{\partial L}{\partial x} = F$$

and by Euler-Lagrange

$$\ddot{x} = \frac{F}{m}$$

Note that

$$\frac{d^2}{dt^2}\left(\frac{Ft^2}{2m} + At + B\right) = \frac{F}{m} = \ddot{x}$$

Hence $x(t)$ is quadratic.

$$x(t) = \frac{Ft^2}{2m} + At + B$$

These are the boundary conditions.

$$x(0) = x_a, \quad x(T) = x_b$$

Solve for B.

$$x(0) = B = x_a$$

Solve for A.

$$x(T) - x(0) = \frac{FT^2}{2m} + AT = x_b - x_a$$

Hence

$$A = \frac{x_b - x_a}{T} - \frac{FT}{2m}$$

Substitute A and B into $x(t)$.

$$x(t) = \frac{Ft^2}{2m} + \frac{(x_b - x_a)t}{T} - \frac{FTt}{2m} + x_a \tag{1}$$

Take the time derivative of $x(t)$ to obtain $\dot{x}(t)$.

$$\dot{x}(t) = \frac{d}{dt}x(t) = \frac{Ft}{m} + \frac{x_b - x_a}{T} - \frac{FT}{2m} \tag{2}$$

The action is

$$
\begin{aligned}
S &= \int_0^T L\, dt \\
&= \int_0^T \left(\frac{m\dot{x}^2}{2} + Fx \right) dt \\
&= \frac{m(x_b - x_a)^2}{2T} + \frac{FT(x_b + x_a)}{2} - \frac{F^2T^3}{24m}
\end{aligned}
\tag{3}
$$