Coherent state

Let $|\Psi\rangle$ be the following "coherent" state where $\bar{n}$ is the mean number of photons and $|n\rangle$ is the state with exactly n photons.

$$|\Psi\rangle = \sum_{n=0}^{\infty} \sqrt{\frac{\bar{n}^n \exp(-\bar{n})}{n!}} \exp\Big(-i\big(n + \tfrac{1}{2}\big)\omega t\Big)|n\rangle$$

Let $\hat{a}$ be the following "lowering" operator.

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$$

Apply operator $\hat{a}$ to coherent state $|\Psi\rangle$ to obtain (see derivation below)

$$\hat{a}|\Psi\rangle = \sqrt{\bar{n}}\exp(-i\omega t)|\Psi\rangle$$

and

$$\langle\Psi|\hat{a}^\dagger = (\hat{a}|\Psi\rangle)^\dagger = \sqrt{\bar{n}}\exp(i\omega t)\langle\Psi|$$

Let $\hat{E}$ be the following electric field operator.

$$\hat{E} = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}(\hat{a} - \hat{a}^\dagger)$$

Note that $\hat{E}$ is Hermitian.

$$\hat{E} = \hat{E}^\dagger$$

Hermitian operators have real eigenvalues, hence $\hat{E}$ corresponds to an observable quantity.

The expected electric field is

$$\langle\hat{E}\rangle = \langle\Psi|\hat{E}|\Psi\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}\langle\Psi|(\hat{a} - \hat{a}^\dagger)|\Psi\rangle$$

By distributive law

$$\langle\hat{E}\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}\left(\langle\Psi|\hat{a}|\Psi\rangle - \langle\Psi|\hat{a}^\dagger|\Psi\rangle\right)$$

Substitute eigenvalues for operators.

$$\langle\hat{E}\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}\left(\sqrt{\bar{n}}\exp(-i\omega t)\langle\Psi|\Psi\rangle - \sqrt{\bar{n}}\exp(i\omega t)\langle\Psi|\Psi\rangle\right)$$

By $\langle\Psi|\Psi\rangle = 1$ we have

$$\langle\hat{E}\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}\left(\sqrt{\bar{n}}\exp(-i\omega t) - \sqrt{\bar{n}}\exp(i\omega t)\right)$$

Recalling that

$$2\sin(\omega t) = i\exp(-i\omega t) - i\exp(i\omega t)$$

we have

$$\langle \hat{E} \rangle = \sqrt{\frac{2\bar{n}\hbar\omega}{\epsilon_0}} \sin(\omega t)$$

Let $\hat{B}$ be the following magnetic field operator.

$$\hat{B} = \sqrt{\frac{\hbar\omega\mu_0}{2}} (\hat{a} + \hat{a}^\dagger)$$

By deduction similar to that for $\langle \hat{E} \rangle$ we obtain

$$\langle \hat{B} \rangle = \sqrt{2\bar{n}\hbar\omega\mu_0} \cos(\omega t)$$

The energy of an electromagnetic wave is

$$U = \frac{\epsilon_0}{2} |\mathbf{E}|^2 + \frac{1}{2\mu_0} |\mathbf{B}|^2$$

For linear polarization there exists a rotation matrix R such that

$$R\mathbf{E} = \begin{pmatrix} E \\ 0 \\ 0 \end{pmatrix}, \quad R\mathbf{B} = \begin{pmatrix} 0 \\ B \\ 0 \end{pmatrix}$$

Hence in the rotated frame

$$U = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2$$

For a quantum field we have

$$U = \frac{\epsilon_0}{2} \langle \hat{E}^2 \rangle + \frac{1}{2\mu_0} \langle \hat{B}^2 \rangle$$

where

$$\begin{aligned} \langle \hat{E}^2 \rangle &= \langle \Psi | \hat{E}\hat{E} | \Psi \rangle = -\frac{\hbar\omega}{2\epsilon_0} \langle \Psi | (\hat{a} - \hat{a}^\dagger)(\hat{a} - \hat{a}^\dagger) | \Psi \rangle \\ \langle \hat{B}^2 \rangle &= \langle \Psi | \hat{B}\hat{B} | \Psi \rangle = \frac{\hbar\omega\mu_0}{2} \langle \Psi | (\hat{a} + \hat{a}^\dagger)(\hat{a} + \hat{a}^\dagger) | \Psi \rangle \end{aligned}$$

For the coherent state

$$\begin{aligned} \langle \Psi | \hat{a}\hat{a} | \Psi \rangle &= \left(\sqrt{\bar{n}} \exp(-i\omega t)\right)^2 && = \bar{n} \exp(-2i\omega t) \\ \langle \Psi | \hat{a}^\dagger \hat{a} | \Psi \rangle &= \left(\sqrt{\bar{n}} \exp(i\omega t)\right) \left(\sqrt{\bar{n}} \exp(-i\omega t)\right) && = \bar{n} \\ \langle \Psi | \hat{a}\hat{a}^\dagger | \Psi \rangle &= \langle \Psi | (\hat{a}^\dagger \hat{a} + 1) | \Psi \rangle = \langle \Psi | a^\dagger \hat{a} | \Psi \rangle + \langle \Psi | \Psi \rangle && = \bar{n} + 1 \\ \langle \Psi | \hat{a}^\dagger \hat{a}^\dagger | \Psi \rangle &= \left(\sqrt{\bar{n}} \exp(i\omega t)\right)^2 && = \bar{n} \exp(2i\omega t) \end{aligned}$$

The expectation $\bar{n} + 1$ for $\hat{a}\hat{a}^\dagger$ is from the commutator

$$\hat{a}\hat{a}^\dagger - \hat{a}^\dagger \hat{a} = 1$$

Using the expectation values derived above we now have

$$\langle \hat{E}^2 \rangle = -\frac{\hbar\omega}{2\epsilon_0}\big(\bar{n}\exp(-2i\omega t) + \bar{n}\exp(2i\omega t) - 2\bar{n} - 1\big)$$
$$\langle \hat{B}^2 \rangle = \frac{\hbar\omega\mu_0}{2}\big(\bar{n}\exp(-2i\omega t) + \bar{n}\exp(2i\omega t) + 2\bar{n} + 1\big)$$

Noting that

$$-4\sin^2(\omega t) = \exp(-2i\omega t) + \exp(2i\omega t) - 2$$
$$4\cos^2(\omega t) = \exp(-2i\omega t) + \exp(2i\omega t) + 2$$

we have

$$\langle \hat{E}^2 \rangle = -\frac{\hbar\omega}{2\epsilon_0}\left(-4\bar{n}\sin^2(\omega t) - 1\right)$$
$$\langle \hat{B}^2 \rangle = \frac{\hbar\omega\mu_0}{2}\left(4\bar{n}\cos^2(\omega t) + 1\right)$$

Rewrite as

$$\frac{\epsilon_0}{2}\langle \hat{E}^2 \rangle = \hbar\omega\left(\bar{n}\sin^2(\omega t) + \tfrac{1}{4}\right) \tag{1}$$
$$\frac{1}{2\mu_0}\langle \hat{B}^2 \rangle = \hbar\omega\left(\bar{n}\cos^2(\omega t) + \tfrac{1}{4}\right) \tag{2}$$

The total energy per unit volume is the sum of (1) and (2).

$$U = \frac{\epsilon_0}{2}\langle \hat{E}^2 \rangle + \frac{1}{2\mu_0}\langle \hat{B}^2 \rangle = \hbar\omega\left(\bar{n} + \tfrac{1}{2}\right)$$

Check units.

$$\hbar\omega = h\nu \propto \text{joule second} \times \frac{1}{\text{second}} = \text{joule}$$

We will now show that

$$\hat{a}|\Psi\rangle = \sqrt{\bar{n}}\exp(-i\omega t)|\Psi\rangle$$

Let

$$c_n = \sqrt{\frac{\bar{n}^n \exp(-\bar{n})}{n!}}\exp\Big(-i\big(n + \tfrac{1}{2}\big)\omega t\Big)$$

It follows that

$$c_n = \sqrt{\frac{\bar{n}}{n}}\exp(-i\omega t)c_{n-1}$$

Hence

$$\hat{a}\big(c_n|n\rangle\big) = c_n\sqrt{n}|n-1\rangle = \sqrt{\bar{n}}\exp(-i\omega t)c_{n-1}|n-1\rangle \tag{3}$$

Noting that $\hat{a}|0\rangle = 0$ we can write the summation starting from $n = 1$.

$$\hat{a}|\Psi\rangle = \hat{a}\sum_{n=1}^{\infty} c_n|n\rangle$$

By equation (3) we have

$$\hat{a}|\Psi\rangle = \sqrt{\bar{n}}\exp(-i\omega t)\sum_{n=1}^{\infty} c_{n-1}|n-1\rangle = \sqrt{\bar{n}}\exp(-i\omega t)|\Psi\rangle$$