Classic integration trick

Show that

$$\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2\sigma^2}\right) \cos(kx)\, dx = \sqrt{2\pi\sigma^2} \exp\left(-\frac{k^2\sigma^2}{2}\right)$$

Start with the following integral.

$$f(k) = \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2\sigma^2}\right) \cos(kx)\, dx + i \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2\sigma^2}\right) \sin(kx)\, dx \tag{1}$$

Rewrite as

$$f(k) = \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2\sigma^2} + ikx\right)\, dx$$

Consider the following well-known Gaussian integral.

$$\int_{-\infty}^{\infty} \exp(-ax^2 + bx)\, dx = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right)$$

For $a = 1/(2\sigma^2)$ and $b = ik$ we have

$$f(k) = \sqrt{2\pi\sigma^2} \exp\left(-\frac{k^2\sigma^2}{2}\right) \tag{2}$$

By equality of the real parts of (1) and (2) we have

$$\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2\sigma^2}\right) \cos(kx)\, dx = \sqrt{2\pi\sigma^2} \exp\left(-\frac{k^2\sigma^2}{2}\right)$$