Born approximation 3

We will now show why we can use the approximation

$$\frac{\exp(ik|\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|}V(\mathbf{y}) \approx \frac{1}{|\mathbf{x}|}\exp\left(ik|\mathbf{x}| - \frac{ik\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}\right)V(\mathbf{y})$$

For scattering experiments we have $\mathbf{x}$ far from the origin where $V(\mathbf{x}) \approx 0$. Hence we can restrict our attention to the $\mathbf{y}$-ball such that $|\mathbf{y}| \ll |\mathbf{x}|$ and let $V(\mathbf{y})$ censor everything else.

Consider the following expansions.

$$|\mathbf{x}-\mathbf{y}|^2 = |\mathbf{x}|^2 - 2\mathbf{x}\cdot\mathbf{y} + |\mathbf{y}|^2$$

$$\left(|\mathbf{x}| - \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}\right)^2 = |\mathbf{x}|^2 - 2\mathbf{x}\cdot\mathbf{y} + \left(\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}\right)^2$$

For $|\mathbf{x}|^2 \gg |\mathbf{y}|^2$ we have

$$|\mathbf{x}-\mathbf{y}|^2 \approx \left(|\mathbf{x}| - \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}\right)^2$$

Noting that $|\mathbf{x}|^2 > \mathbf{x}\cdot\mathbf{y}$ we have by positivity

$$|\mathbf{x}-\mathbf{y}| \approx |\mathbf{x}| - \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}$$

Hence

$$\exp(ik|\mathbf{x}-\mathbf{y}|) \approx \exp\left(ik|\mathbf{x}| - \frac{ik\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}\right) \tag{1}$$

For the factor

$$\frac{1}{|\mathbf{x}-\mathbf{y}|}$$

we discard $\mathbf{y}$ altogether and use

$$\frac{1}{|\mathbf{x}-\mathbf{y}|} \approx \frac{1}{|\mathbf{x}|} \tag{2}$$

Hence by (1) and (2) we have

$$\frac{\exp(ik|\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|}V(\mathbf{y}) \approx \frac{1}{|\mathbf{x}|}\exp\left(ik|\mathbf{x}| - \frac{ik\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}\right)V(\mathbf{y})$$