Born approximation 2

We will now derive the Green's function that was used in the previous section.

Let $G(\mathbf{x})$ be a Green's function such that

$$\left(\nabla^2 + k^2\right) G(\mathbf{x}) = \delta(\mathbf{x}) \tag{1}$$

Let $g(\mathbf{y})$ be the Fourier transform of $G(\mathbf{x})$ such that

$$G(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int \exp(i\mathbf{x}\cdot\mathbf{y}) g(\mathbf{y})\, d\mathbf{y} \tag{2}$$

By equations (1) and (2)

$$\left(\nabla^2 + k^2\right) \left[\frac{1}{(2\pi)^{3/2}} \int \exp(i\mathbf{x}\cdot\mathbf{y}) g(\mathbf{y})\, d\mathbf{y}\right] = \delta(\mathbf{x})$$

By linearity

$$\frac{1}{(2\pi)^{3/2}} \int \left(\nabla^2 + k^2\right) \exp(i\mathbf{x}\cdot\mathbf{y}) g(\mathbf{y})\, d\mathbf{y} = \delta(\mathbf{x})$$

Noting that ∇^2 applies $\mathbf{x}$ and not $\mathbf{y}$ we have

$$\nabla^2 \exp(i\mathbf{x}\cdot\mathbf{y}) g(\mathbf{y}) = -|\mathbf{y}|^2 \exp(i\mathbf{x}\cdot\mathbf{y}) g(\mathbf{y})$$

Hence

$$\frac{1}{(2\pi)^{3/2}} \int \left(-|\mathbf{y}|^2 + k^2\right) \exp(i\mathbf{x}\cdot\mathbf{y}) g(\mathbf{y})\, d\mathbf{y} = \delta(\mathbf{x})$$

By the identity

$$\delta(\mathbf{x}) = \frac{1}{(2\pi)^3} \int \exp(i\mathbf{x}\cdot\mathbf{y})\, d\mathbf{y}$$

we have

$$\frac{1}{(2\pi)^{3/2}} \int \left(-|\mathbf{y}|^2 + k^2\right) \exp(i\mathbf{x}\cdot\mathbf{y}) g(\mathbf{y})\, d\mathbf{y} = \frac{1}{(2\pi)^3} \int \exp(i\mathbf{x}\cdot\mathbf{y})\, d\mathbf{y}$$

Hence

$$g(\mathbf{y}) = \frac{1}{(2\pi)^{3/2} \left(k^2 - |\mathbf{y}|^2\right)}$$

Substitute for $g(\mathbf{y})$ in (2) to obtain

$$G(\mathbf{x}) = \frac{1}{(2\pi)^3} \int \frac{\exp(i\mathbf{x}\cdot\mathbf{y})}{k^2 - |\mathbf{y}|^2}\, d\mathbf{y}$$

Change to polar coordinates where $x = |\mathbf{x}|$ and $r = |\mathbf{y}|$.

$$G(\mathbf{x}) = \frac{1}{(2\pi)^3} \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{\exp(ixr\cos\theta)}{k^2 - r^2}\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Integrate over ϕ (multiply by 2π).

$$G(\mathbf{x})=\frac{1}{(2\pi)^2}\int_0^\infty\int_0^\pi\frac{\exp(ixr\cos\theta)}{k^2-r^2}\,r^2\sin\theta\,dr\,d\theta$$

For the integral over θ use change of variable $u=\cos\theta$ and $du=-\sin\theta\,d\theta$ to obtain

$$G(\mathbf{x})=\frac{1}{(2\pi)^2}\int_0^\infty\int_1^{-1}-\frac{\exp(ixru)}{k^2-r^2}\,r^2\,dr\,du$$

By the integral

$$\int_1^{-1}-\exp(ixru)\,du=\frac{2\sin(xr)}{xr}\tag{3}$$

we have

$$G(\mathbf{x})=\frac{1}{2\pi^2x}\int_0^\infty\frac{\sin(xr)}{k^2-r^2}\,r\,dr$$

Noting that $r\sin(xr)$ is an even function of r we can extend the lower limit as

$$G(\mathbf{x})=\frac{1}{4\pi^2x}\int_{-\infty}^\infty\frac{\sin(xr)}{k^2-r^2}\,r\,dr$$

Negate the denominator.

$$G(\mathbf{x})=\frac{1}{4\pi^2x}\int_{-\infty}^\infty-\frac{\sin(xr)}{r^2-k^2}\,r\,dr$$

Change the sine function to exponential form and factor the denominator.

$$G(\mathbf{x})=\frac{i}{8\pi^2x}\left[\int_{-\infty}^\infty\frac{\exp(ixr)}{(r+k)(r-k)}\,r\,dr-\int_{-\infty}^\infty\frac{\exp(-ixr)}{(r+k)(r-k)}\,r\,dr\right]\tag{4}$$

By Cauchy's integral formula we have

$$\int_{-\infty}^\infty\frac{r\exp(ixy)}{r+k}\frac{1}{r-k}\,dr=i\pi\exp(ikx)$$

and

$$\int_{-\infty}^\infty\frac{r\exp(-ixy)}{r-k}\frac{1}{r+k}\,dr=-i\pi\exp(ikx)$$

Hence

$$G(\mathbf{x})=-\frac{\exp(ikx)}{4\pi x}\tag{5}$$

where

$$x=|\mathbf{x}|$$

Eigenmath script