Born approximation 1

Let $\phi(\mathbf{p})$ be the scattering amplitude for momentum transfer vector $\mathbf{p}$ where

$$\mathbf{p} = \mathbf{p}_i - \mathbf{p}_f$$

The Born approximation for $\phi(\mathbf{p})$ is

$$\phi(\mathbf{p}) = -\frac{m}{2\pi\hbar^2} \int \exp\left(\frac{i\mathbf{p}\cdot\mathbf{y}}{\hbar}\right) V(\mathbf{y})\, d\mathbf{y}$$

For example, given Coulomb potential

$$V(\mathbf{y}) = -\frac{Ze^2}{4\pi\varepsilon_0 |\mathbf{y}|} = -\frac{Z\alpha\hbar c}{|\mathbf{y}|}$$

the solution is

$$\phi(\mathbf{p}) = -\frac{2mZ\alpha\hbar c}{|\mathbf{p}|^2}$$

By the identity

$$|\mathbf{p}|^2 = 4mE(1-\cos\theta)$$

we have

$$\phi(\theta) = -\frac{Z\alpha\hbar c}{2E(1-\cos\theta)}$$

Hence the Rutherford scattering cross section

$$\frac{d\sigma}{d\Omega} = |\phi(\theta)|^2 = \left[\frac{Z\alpha\hbar c}{2E(1-\cos\theta)}\right]^2 \tag{1}$$

Let us derive $\phi(\mathbf{p})$ from the time-independent Schrodinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{x}) + V(\mathbf{x})\psi(\mathbf{x}) = E\psi(\mathbf{x})$$

Rewrite the Schrodinger equation as

$$\left(\nabla^2 + k^2\right)\psi(\mathbf{x}) = \frac{2m}{\hbar^2}V(\mathbf{x})\psi(\mathbf{x})$$

where

$$k = \frac{\sqrt{2mE}}{\hbar}$$

Now we are going to make the Schrodinger equation even more complicated.

By the sifting property of $\delta(\mathbf{x}-\mathbf{y})$ we can write

$$\left(\nabla^2 + k^2\right)\psi(\mathbf{x}) = \frac{2m}{\hbar^2}\int \delta(\mathbf{x}-\mathbf{y})V(\mathbf{y})\psi(\mathbf{y})\, d\mathbf{y}$$

Let $G(\mathbf{x})$ be a Green's function such that

$$\left(\nabla^2 + k^2\right) G(\mathbf{x}) = \delta(\mathbf{x})$$

Then by substitution

$$\left(\nabla^2 + k^2\right) \psi(\mathbf{x}) = \frac{2m}{\hbar^2} \int \left(\nabla^2 + k^2\right) G(\mathbf{x} - \mathbf{y}) V(\mathbf{y}) \psi(\mathbf{y})\, d\mathbf{y}$$

By linearity

$$\left(\nabla^2 + k^2\right) \psi(\mathbf{x}) = \left(\nabla^2 + k^2\right) \frac{2m}{\hbar^2} \int G(\mathbf{x} - \mathbf{y}) V(\mathbf{y}) \psi(\mathbf{y})\, d\mathbf{y}$$

Hence

$$\psi(\mathbf{x}) = \frac{2m}{\hbar^2} \int G(\mathbf{x} - \mathbf{y}) V(\mathbf{y}) \psi(\mathbf{y})\, d\mathbf{y}$$

The solution for $G(\mathbf{x})$ is

$$G(\mathbf{x}) = -\frac{\exp(ik|\mathbf{x}|)}{4\pi|\mathbf{x}|}$$

Hence

$$\psi(\mathbf{x}) = -\frac{m}{2\pi\hbar^2} \int \frac{\exp(ik|\mathbf{x} - \mathbf{y}|)}{|\mathbf{x} - \mathbf{y}|} V(\mathbf{y}) \psi(\mathbf{y})\, d\mathbf{y}$$

For scattering experiments we can use the approximation

$$\frac{\exp(ik|\mathbf{x} - \mathbf{y}|)}{|\mathbf{x} - \mathbf{y}|} V(\mathbf{y}) \approx \frac{1}{|\mathbf{x}|} \exp\left(ik|\mathbf{x}| - \frac{ik\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|}\right) V(\mathbf{y})$$

Hence

$$\psi(\mathbf{x}) = -\frac{m}{2\pi\hbar^2} \frac{\exp(ik|\mathbf{x}|)}{|\mathbf{x}|} \int \exp\left(-\frac{ik\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|}\right) V(\mathbf{y}) \psi(\mathbf{y})\, d\mathbf{y}$$

For the Born approximation we use

$$\psi(\mathbf{y}) = \exp\left(\frac{ik\mathbf{x}' \cdot \mathbf{y}}{|\mathbf{x}'|}\right)$$

Hence

$$\psi(\mathbf{x}) = -\frac{m}{2\pi\hbar^2} \frac{\exp(ik|\mathbf{x}|)}{|\mathbf{x}|} \int \exp\left(\frac{ik\mathbf{x}' \cdot \mathbf{y}}{|\mathbf{x}'|} - \frac{ik\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|}\right) V(\mathbf{y})\, d\mathbf{y}$$

Let $\mathbf{p}$ be momentum transfer

$$\mathbf{p} = \hbar k \left(\frac{\mathbf{x}'}{|\mathbf{x}'|} - \frac{\mathbf{x}}{|\mathbf{x}|}\right)$$

Then by substitution

$$\psi(\mathbf{x}) = -\frac{m}{2\pi\hbar^2} \frac{\exp(ik|\mathbf{x}|)}{|\mathbf{x}|} \int \exp\left(\frac{i\mathbf{p} \cdot \mathbf{y}}{\hbar}\right) V(\mathbf{y})\, d\mathbf{y}$$

Partition $\psi(\mathbf{x})$ as

$$\psi(\mathbf{x}) = \frac{\exp(ik|\mathbf{x}|)}{|\mathbf{x}|} \phi(\mathbf{p})$$

where $\phi(\mathbf{p})$ is the scattering amplitude

$$\phi(\mathbf{p}) = -\frac{m}{2\pi\hbar^2} \int \exp\left(\frac{i\mathbf{p}\cdot\mathbf{y}}{\hbar}\right) V(\mathbf{y})\, d\mathbf{y}$$

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