Bloch sphere

Let $|s\rangle$ be the spin state

$$|s\rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)\exp(i\phi) \end{pmatrix}$$

The expectation values for $|s\rangle$ are

$$\langle S_x \rangle = \langle s|S_x|s\rangle = \frac{\hbar}{2}\sin\theta\cos\phi$$

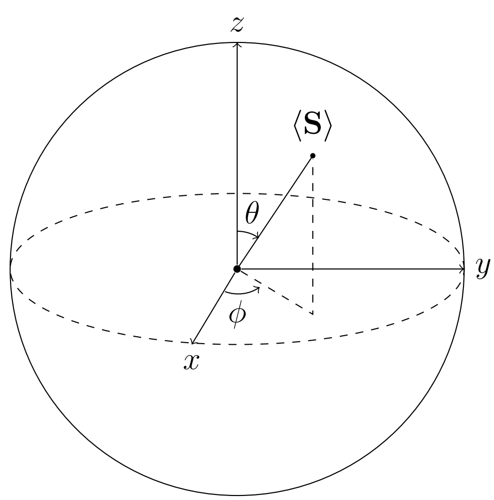
$$\langle S_y \rangle = \langle s|S_y|s\rangle = \frac{\hbar}{2}\sin\theta\sin\phi$$

$$\langle S_z \rangle = \langle s|S_z|s\rangle = \frac{\hbar}{2}\cos\theta$$

In vector form

$$\langle \mathbf{S} \rangle = \langle s|\mathbf{S}|s\rangle = \frac{\hbar}{2}\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

In spherical coordinates $\langle \mathbf{S} \rangle = (\hbar/2, \theta, \phi)$. Hence $\langle \mathbf{S} \rangle$ is a point on a sphere of radius $\hbar/2$.



To show completeness, let $|\chi\rangle$ be a vector in $\mathbb{C}^2$ such that $\langle\chi|\chi\rangle = 1$.

$$|\chi\rangle = \begin{pmatrix} A\exp(i\phi_1) \\ \sqrt{1-A^2}\exp(i\phi_2) \end{pmatrix} \qquad 0 \le A \le 1$$

Substitute $\cos(\theta/2)$ for A and $\sin(\theta/2)$ for $\sqrt{1-A^2}$.

$$|\chi\rangle = \begin{pmatrix} \cos(\theta/2)\exp(i\phi_1) \\ \sin(\theta/2)\exp(i\phi_2) \end{pmatrix} \qquad 0 \le \theta \le \pi$$

Spin $|s\rangle$ is a phase of $|\chi\rangle$ with $\phi = \phi_2 - \phi_1$.

$$|s\rangle = \exp(-i\phi_1)|\chi\rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)\exp(i\phi) \end{pmatrix}$$

By equality of expectation every $|\chi\rangle$ has a place on the Bloch sphere.

$$\langle \mathbf{S} \rangle = \langle \chi | \mathbf{S} | \chi \rangle = \langle s | \mathbf{S} | s \rangle$$

Eigenmath script