

Bhabha scattering 2

Here we solve path integrals to find $\mathcal{M}_1$ and $\mathcal{M}_2$ that were used in the previous section.

$\mathcal{M}_1$ is the amplitude for $1 \rightarrow 3, 4 \rightarrow 2$.

$$\mathcal{M}_1 = i \iint \bar{\psi}_2(y)(-ie\gamma^\mu)\psi_4(y)G(x,y)\bar{\psi}_3(x)(-ie\gamma_\mu)\psi_1(x)d^4x d^4y$$

Function $G(x, y)$ is the photon propagator

$$G(x, y) = \frac{i}{(2\pi)^4} \int \frac{\exp(-ip \cdot (x - y))}{p^2 + i\epsilon} d^4p$$

The normalized wave functions are (V is volume)

$$\begin{aligned}\psi_1(x) &= \sqrt{2EV}u_1 \exp(-ip_1 \cdot x) \\ \psi_2(x) &= \sqrt{2EV}v_2 \exp(ip_2 \cdot x) \\ \psi_3(x) &= \sqrt{2EV}u_3 \exp(-ip_3 \cdot x) \\ \psi_4(x) &= \sqrt{2EV}v_4 \exp(ip_4 \cdot x)\end{aligned}$$

Substitute for the wave functions in $\mathcal{M}_1$ and factor the spinor products.

$$\begin{aligned}\mathcal{M}_1 &= -\frac{e^2}{4E^2V^2}(\bar{v}_2\gamma^\mu v_4)(\bar{u}_3\gamma_\mu u_1) \\ &\quad i \iint \exp(i(-p_2 + p_4) \cdot y) G(x, y) \exp(i(p_3 - p_1) \cdot x) d^4x d^4y\end{aligned}$$

Substitute for $G(x, y)$.

$$\begin{aligned}\mathcal{M}_1 &= \frac{e^2}{4E^2V^2}(\bar{v}_2\gamma^\mu v_4)(\bar{u}_3\gamma_\mu u_1) \frac{1}{(2\pi)^4} \\ &\quad \iiint \exp(i(-p_2 + p_4) \cdot y) \frac{\exp(-ip \cdot (x - y))}{p^2 + i\epsilon} \exp(i(p_3 - p_1) \cdot x) d^4x d^4y d^4p\end{aligned}$$

Combine exponentials.

$$\begin{aligned}\mathcal{M}_1 &= \frac{e^2}{4E^2V^2}(\bar{v}_2\gamma^\mu v_4)(\bar{u}_3\gamma_\mu u_1) \frac{1}{(2\pi)^4} \\ &\quad \iiint \frac{\exp(i(-p_2 + p_4 + p) \cdot y) \exp(i(p_3 - p_1 - p) \cdot x)}{p^2 + i\epsilon} d^4x d^4y d^4p\end{aligned}$$

Solve the integral over x .

$$\int \exp(i(p_3 - p_1 - p) \cdot x) d^4x = (2\pi)^4 \delta^4(p_3 - p_1 - p)$$

Solve the integral over y .

$$\int \exp(i(-p_2 + p_4 + p) \cdot y) d^4y = (2\pi)^4 \delta^4(-p_2 + p_4 + p)$$

Hence

$$\mathcal{M}_1 = \frac{e^2}{4E^2V^2} (\bar{v}_2 \gamma^\mu v_4) (\bar{u}_3 \gamma_\mu u_1) (2\pi)^4 \int \frac{\delta^4(-p_2 + p_4 + p) \delta^4(p_3 - p_1 - p)}{p^2 + i\epsilon} d^4p$$

The integrand vanishes except for

$$p = p_2 - p_4 = p_3 - p_1$$

Hence

$$\mathcal{M}_1 = \frac{e^2}{4E^2V^2} (\bar{v}_2 \gamma^\mu v_4) (\bar{u}_3 \gamma_\mu u_1) (2\pi)^4 \frac{\delta^4(p_4 + p_3 - p_2 - p_1)}{(p_3 - p_1)^2}$$

$\mathcal{M}_2$ is the amplitude for $1 \rightarrow 2, 4 \rightarrow 3$.

$$\mathcal{M}_2 = i \iint \bar{\psi}_3(y) (-ie\gamma^\mu) \psi_4(y) G(x, y) \bar{\psi}_2(x) (-ie\gamma_\mu) \psi_1(x) d^4x d^4y$$

It follows that

$$\mathcal{M}_2 = \frac{e^2}{4E^2V^2} (\bar{u}_3 \gamma^\mu v_4) (\bar{v}_2 \gamma_\mu u_1) (2\pi)^4 \int \frac{\delta^4(p_3 + p_4 + p) \delta^4(-p_2 - p_1 - p)}{p^2 + i\epsilon} d^4p$$

The integrand vanishes except for

$$p = -p_3 - p_4 = -p_1 - p_2$$

Hence

$$\mathcal{M}_2 = \frac{e^2}{4E^2V^2} (\bar{u}_3 \gamma^\mu v_4) (\bar{v}_2 \gamma_\mu u_1) (2\pi)^4 \frac{\delta^4(p_4 + p_3 - p_2 - p_1)}{(p_1 + p_2)^2}$$

For the momentum vectors given in the previous section we have

$$\delta^4(p_4 + p_3 - p_2 - p_1) = \delta^4(0) = 1$$

Hence

$$\begin{aligned} \mathcal{M}_1 &= \frac{(2\pi)^4}{4E^2V^2} \frac{e^2}{(p_3 - p_1)^2} (\bar{v}_2 \gamma^\mu v_4) (\bar{u}_3 \gamma_\mu u_1) \\ \mathcal{M}_2 &= \frac{(2\pi)^4}{4E^2V^2} \frac{e^2}{(p_1 + p_2)^2} (\bar{u}_3 \gamma^\mu v_4) (\bar{v}_2 \gamma_\mu u_1) \end{aligned}$$

In practice the leading coefficient is absorbed in the calculation of $d\sigma$ hence we use the formulas

$$\begin{aligned} \mathcal{M}_1 &= \frac{e^2}{(p_3 - p_1)^2} (\bar{v}_2 \gamma^\mu v_4) (\bar{u}_3 \gamma_\mu u_1) \\ \mathcal{M}_2 &= \frac{e^2}{(p_1 + p_2)^2} (\bar{u}_3 \gamma^\mu v_4) (\bar{v}_2 \gamma_\mu u_1) \end{aligned}$$

Eigenmath script