Atomic transitions 7

From the previous section the spontaneous emission rate is

$$A_{b\to a} = \frac{e^2}{3\pi\varepsilon_0\hbar c^3}\omega_{ba}^3\left|\langle\psi_a|\mathbf{r}|\psi_b\rangle\right|^2$$

For hydrogen $2p \to 1s$ we have

$$A_{21} = \frac{e^2}{3\pi\varepsilon_0\hbar c^3}\omega_{21}^3\left|\langle\psi_{100}|\mathbf{r}|\psi_{210}\rangle\right|^2$$

Noting that

$$e^2 = 4\pi\varepsilon_0\hbar c\alpha$$

we can also write

$$A_{21} = \frac{4\alpha}{3c^2}\omega_{21}^3\left|\langle\psi_{100}|\mathbf{r}|\psi_{210}\rangle\right|^2 \tag{1}$$

Verify dimensions:

$$A_{21} \propto (\mathrm{m/s})^{-2} \times \mathrm{s}^{-3} \times \mathrm{m}^2 = \mathrm{s}^{-1} \text{ (or hertz)}$$

For angular frequency ω_{21} we have

$$\omega_{21} = \frac{E_2 - E_1}{\hbar}$$

and for hydrogen

$$E_n = -\frac{\alpha\hbar c}{2n^2 a_0}$$

Hence

$$\omega_{21} = \frac{3\alpha c}{8a_0}$$

For the transition "probability" we have

$$\left|\langle\psi_{100}|\mathbf{r}|\psi_{210}\rangle\right|^2 = |x_{21}|^2 + |y_{21}|^2 + |z_{21}|^2$$

where

$$x_{21} = \int_0^\infty\int_0^\pi\int_0^{2\pi} \psi_{100}^*\, x\, \psi_{210}\, dV$$

$$y_{21} = \int_0^\infty\int_0^\pi\int_0^{2\pi} \psi_{100}^*\, y\, \psi_{210}\, dV$$

$$z_{21} = \int_0^\infty\int_0^\pi\int_0^{2\pi} \psi_{100}^*\, z\, \psi_{210}\, dV$$

and

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta, \quad dV = r^2\sin\theta\, dr\, d\theta\, d\phi$$

The integrals work out to be

$$x_{21} = 0, \quad y_{21} = 0, \quad z_{21} = \frac{2^7}{3^5}\sqrt{2}a_0$$

hence

$$\left|\langle\psi_{100}|\mathbf{r}|\psi_{210}\rangle\right|^2 = |z_{21}|^2 = \frac{2^{15}}{3^{10}}a_0^2$$

By equation (1) the spontaneous emission rate is

$$A_{21} = \frac{2^8}{3^8}\frac{\alpha^4 c}{a_0} = 6.26 \times 10^8\,\mathrm{s}^{-1}$$

Noting that

$$a_0 = \frac{\hbar}{\alpha\mu c}$$

we can also write

$$A_{21} = \frac{2^8}{3^8}\frac{\alpha^5\mu c^2}{\hbar} = 6.26 \times 10^8\,\mathrm{s}^{-1}$$

where μ is reduced electron mass

$$\mu = \frac{m_e m_p}{m_e + m_p}$$