Atomic transitions 2

For electric field

$$\mathbf{E} = E_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t)\boldsymbol{\epsilon}$$

the Hamiltonian is

$$H_1(\mathbf{r},t) = -\frac{eE_0}{m\omega}\cos(\mathbf{k}\cdot\mathbf{r} - \omega t)\boldsymbol{\epsilon}\cdot\mathbf{p}$$

In exponential form

$$H_1(\mathbf{r},t) = -\frac{eE_0}{m\omega}\left(\tfrac{1}{2}\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t) + \tfrac{1}{2}\exp(-i\mathbf{k}\cdot\mathbf{r} + i\omega t)\right)\boldsymbol{\epsilon}\cdot\mathbf{p}$$

Given the initial condition $c_b(0) = 0$ the first-order approximation for $c_b(t)$ is

$$c_b(t) = -\frac{i}{\hbar}\int_0^t \langle\psi_b|H_1(\mathbf{r},t')|\psi_a\rangle \exp(i\omega_0 t')\,dt', \quad \omega_0 = \frac{E_b - E_a}{\hbar}$$

Factor the integrand.

$$\begin{aligned} c_b(t) = \frac{ieE_0}{2m\hbar\omega}\langle\psi_b|\exp(i\mathbf{k}\cdot\mathbf{r})\boldsymbol{\epsilon}\cdot\mathbf{p}|\psi_a\rangle \int_0^t \exp(-i\omega t')\exp(i\omega_0 t')\,dt' \\ + \frac{ieE_0}{2m\hbar\omega}\langle\psi_b|\exp(-i\mathbf{k}\cdot\mathbf{r})\boldsymbol{\epsilon}\cdot\mathbf{p}|\psi_a\rangle \int_0^t \exp(i\omega t')\exp(i\omega_0 t')\,dt' \end{aligned}$$

Solve the time integrals to obtain

$$\begin{aligned} c_b(t) = \frac{eE_0}{2m\hbar\omega}\langle\psi_b|\exp(i\mathbf{k}\cdot\mathbf{r})\boldsymbol{\epsilon}\cdot\mathbf{p}|\psi_a\rangle \frac{\exp\big(i(\omega_0-\omega)t\big)-1}{\omega_0-\omega} \\ + \frac{eE_0}{2m\hbar\omega}\langle\psi_b|\exp(-i\mathbf{k}\cdot\mathbf{r})\boldsymbol{\epsilon}\cdot\mathbf{p}|\psi_a\rangle \frac{\exp\big(i(\omega_0+\omega)t\big)-1}{\omega_0+\omega} \end{aligned} \tag{1}$$

As an approximation, discard the second term since the first term dominates for $\omega \approx \omega_0$.

$$c_b(t) = \frac{eE_0}{2m\hbar\omega}\langle\psi_b|\exp(i\mathbf{k}\cdot\mathbf{r})\boldsymbol{\epsilon}\cdot\mathbf{p}|\psi_a\rangle \frac{\exp\big(i(\omega_0-\omega)t\big)-1}{\omega_0-\omega}$$

Rewrite $c_b(t)$ in the form of a sine function.

$$c_b(t) = \frac{ieE_0}{m\hbar\omega}\langle\psi_b|\exp(i\mathbf{k}\cdot\mathbf{r})\boldsymbol{\epsilon}\cdot\mathbf{p}|\psi_a\rangle \frac{\sin\big(\frac{1}{2}(\omega_0-\omega)t\big)}{\omega_0-\omega}\exp\big(\tfrac{i}{2}(\omega_0-\omega)t\big) \tag{2}$$

Verify dimensions.

$$H_1(\mathbf{r},t) \propto \frac{\overset{e}{\mathrm{C}}\ \ \overset{E_0}{\mathrm{N\,C^{-1}}}}{\underset{\mathrm{kg}}{m}\ \ \underset{\mathrm{s^{-1}}}{\omega}} \times \underset{\mathrm{kg\,m\,s^{-1}}}{\boldsymbol{\epsilon}\cdot\mathbf{p}} = \mathrm{N\,m} = \mathrm{J}$$

$$c_b(t) \propto \frac{\overset{e}{\mathrm{C}}\ \ \overset{E_0}{\mathrm{N\,C^{-1}}}}{\underset{\mathrm{kg}}{m}\ \ \underset{\mathrm{J\,s}}{\hbar}\ \ \underset{\mathrm{s^{-1}}}{\omega}} \times \frac{\overset{\langle\psi_b|\exp(i\mathbf{k}\cdot\mathbf{r})\boldsymbol{\epsilon}\cdot\mathbf{p}|\psi_a\rangle}{\mathrm{kg\,m\,s^{-1}}}}{\underset{\mathrm{s^{-1}}}{\omega_0-\omega}} = \frac{\mathrm{N\,m}}{\mathrm{J}} = 1$$

Wave functions ψ_a and ψ_b have dimension meter$^{-1/2}$ hence they cancel with $dx \propto$ meter in the integral leaving units of momentum due to $\mathbf{p}$.