

Associated Legendre polynomials

Verify that

$$(1-x^2)\frac{d^2}{dx^2}P_{lm}(x) - 2x\frac{d}{dx}P_{lm}(x) + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P_{lm}(x) = 0 \quad (1)$$

where $P_{lm}(x)$ are associated Legendre polynomials

$$P_{lm}(x) = \begin{cases} \frac{(-1)^m}{2^l l!} (1-x^2)^{\frac{m}{2}} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l, & m \geq 0 \\ (-1)^m \frac{(l+m)!}{(l-m)!} P_{l|m|}(x), & m < 0 \end{cases}$$

Associated Legendre polynomials are needed for spherical harmonic functions $Y_{lm}(\theta, \phi)$.

$$Y_{lm}(\theta, \phi) = (-1)^m \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_{lm}(\cos\theta) \exp(im\phi)$$

See arxiv.org/abs/1805.12125 for the following form of $P_{lm}(\cos\theta)$.

$$P_{lm}(\cos\theta) = \begin{cases} \left(\frac{\sin\theta}{2}\right)^m \sum_{k=0}^{l-m} (-1)^k \frac{(l+m+k)!}{(l-m-k)!(m+k)!k!} \left(\frac{1-\cos\theta}{2}\right)^k, & m \geq 0 \\ (-1)^m \frac{(l+m)!}{(l-m)!} P_{l|m|}(\cos\theta), & m < 0 \end{cases}$$