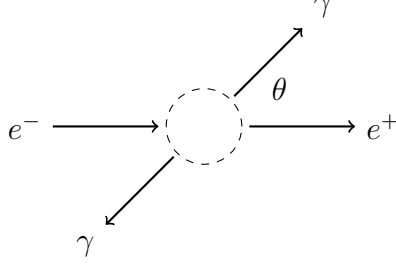


Annihilation 1

Annihilation is the process $e^- + e^+ \rightarrow \gamma + \gamma$. The arrow for e^+ is reversed because a positron is treated mathematically as an electron moving backwards in time.



Momentum vectors where $E = \sqrt{p^2 + m^2}$.

$$p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix}_{e^- \rightarrow} \quad p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix}_{\rightarrow e^+} \quad p_3 = \begin{pmatrix} E \\ E \sin \theta \cos \phi \\ E \sin \theta \sin \phi \\ E \cos \theta \end{pmatrix}_{\nearrow \gamma} \quad p_4 = \begin{pmatrix} E \\ -E \sin \theta \cos \phi \\ -E \sin \theta \sin \phi \\ -E \cos \theta \end{pmatrix}_{\nwarrow \gamma}$$

Spinors for p_1 .

$$u_{11} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix}_{\text{spin up}} \quad u_{12} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix}_{\text{spin down}}$$

Spinors for p_2 .

$$v_{21} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} -p \\ 0 \\ E+m \\ 0 \end{pmatrix}_{\text{spin up}} \quad v_{22} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ p \\ 0 \\ E+m \end{pmatrix}_{\text{spin down}}$$

The scattering amplitude $\mathcal{M}$ is

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$$

where $\mathcal{M}_1$ is the t -channel amplitude

$$\mathcal{M}_1 = \frac{\bar{v}_{2b}(-ie\gamma^\mu)(\not{q}_1 + m)(-ie\gamma^\nu)u_{1a}}{t - m^2}$$

and $\mathcal{M}_2$ is the u -channel amplitude

$$\mathcal{M}_2 = \frac{\bar{v}_{2b}(-ie\gamma^\nu)(\not{q}_2 + m)(-ie\gamma^\mu)u_{1a}}{u - m^2}$$

Matrices $\not{q}_1$ and $\not{q}_2$ are momentum transfer.

$$\begin{aligned}\not{q}_1 &= (p_1 - p_3)^\alpha g_{\alpha\beta} \gamma^\beta \\ \not{q}_2 &= (p_1 - p_4)^\alpha g_{\alpha\beta} \gamma^\beta\end{aligned}$$

Expected density $\langle |\mathcal{M}|^2 \rangle$ is the average density.

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{ab} |\mathcal{M}|^2 = \frac{1}{4} \sum_{ab} (\mathcal{M}_1 + \mathcal{M}_2)(\mathcal{M}_1 + \mathcal{M}_2)^*$$

Expand the summand. (Note indices μ and ν of $\mathcal{M}_2$ are interchanged relative to $\mathcal{M}_1$.)

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{ab} (\mathcal{M}_1^{\mu\nu} \mathcal{M}_{1\mu\nu}^* + \mathcal{M}_1^{\mu\nu} \mathcal{M}_{2\nu\mu}^* + \mathcal{M}_2^{\nu\mu} \mathcal{M}_{1\mu\nu}^* + \mathcal{M}_2^{\nu\mu} \mathcal{M}_{2\nu\mu}^*)$$

The summation results are

$$\begin{aligned}\sum_{ab} \mathcal{M}_1^{\mu\nu} \mathcal{M}_{1\mu\nu}^* &= \frac{e^4}{(t - m^2)^2} (8tu - 24m^2t - 8m^2u - 8m^4) \\ \sum_{ab} \mathcal{M}_1^{\mu\nu} \mathcal{M}_{2\nu\mu}^* &= \frac{e^4}{(t - m^2)(u - m^2)} (8m^2s - 32m^4) \\ \sum_{ab} \mathcal{M}_2^{\nu\mu} \mathcal{M}_{2\nu\mu}^* &= \frac{e^4}{(u - m^2)^2} (8tu - 8m^2t - 24m^2u - 8m^4)\end{aligned}$$

where s , t , and u are Mandelstam variables

$$\begin{aligned}s &= (p_1 + p_2)^2 \\ t &= (p_1 - p_3)^2 \\ u &= (p_1 - p_4)^2\end{aligned}$$

For clarity define

$$\begin{aligned}f_{11} &= 8tu - 24m^2t - 8m^2u - 8m^4 \\ f_{12} &= 8m^2s - 32m^4 \\ f_{22} &= 8tu - 8m^2t - 24m^2u - 8m^4\end{aligned}$$

and write $\langle |\mathcal{M}|^2 \rangle$ more compactly as

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4} \left(\frac{f_{11}}{(t - m^2)^2} + \frac{2f_{12}}{(t - m^2)(u - m^2)} + \frac{f_{22}}{(u - m^2)^2} \right)$$

For $E \gg m$ we use the approximation $m = 0$ and obtain

$$\begin{aligned}f_{11} &= 8tu \\ f_{12} &= 0 \\ f_{22} &= 8tu\end{aligned}$$

Then $\langle |\mathcal{M}|^2 \rangle$ simplifies as

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4} \left(\frac{8tu}{t^2} + \frac{8tu}{u^2} \right) = 2e^4 \left(\frac{u}{t} + \frac{t}{u} \right)$$

For $m = 0$ the Mandelstam variables are

$$\begin{aligned} s &= 4E^2 \\ t &= -2E^2(1 - \cos \theta) \\ u &= -2E^2(1 + \cos \theta) \end{aligned}$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{1 + \cos \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{1 + \cos \theta} \right)$$

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{e^4} \frac{\langle |\mathcal{M}|^2 \rangle}{4s} = \frac{\alpha^2}{8E^2} \left(\frac{1 + \cos \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{1 + \cos \theta} \right)$$

Multiply by $(\hbar c)^2$ to convert from natural units to physical units.

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{8E^2} \left(\frac{1 + \cos \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{1 + \cos \theta} \right) (\hbar c)^2$$

Eigenmath script