Angular momentum 2

The cross product in $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ demands rectangular coordinates. Hence for a wavefunction ψ in spherical coordinates, vectors $\mathbf{r}$ and $\mathbf{p}$ must be transformed in $\mathbf{L}\psi$.

Vector $\mathbf{r}$ transforms as

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{pmatrix}$$

To transform $\mathbf{p}$ we have by the chain rule

$$\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

Hence

$$p_x = -i\hbar\frac{\partial}{\partial x} = -i\hbar\left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$p_y = -i\hbar\frac{\partial}{\partial y} = -i\hbar\left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$p_z = -i\hbar\frac{\partial}{\partial z} = -i\hbar\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)$$

Using the transformed coordinates

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

we have in spherical coordinates

$$L_x = yp_z - zp_y = i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\theta\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$L_y = zp_x - xp_z = i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \frac{\cos\theta\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$L_z = xp_y - yp_x = -i\hbar\frac{\partial}{\partial\phi}$$

and

$$L^2 = {L_x}^2 + {L_y}^2 + {L_z}^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

Eigenmath script

Note

The following construction is from "Introduction to Quantum Mechanics" by Griffiths.

Noting that in spherical coordinates

$$\mathbf{r} = r\mathbf{e}_r$$

and

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

we have for angular momentum $\mathbf{L}$

$$\mathbf{L} = -i\hbar(\mathbf{r}\times\nabla) = -i\hbar\left[r(\mathbf{e}_r\times\mathbf{e}_r)\frac{\partial}{\partial r} + (\mathbf{e}_r\times\mathbf{e}_\theta)\frac{\partial}{\partial\theta} + (\mathbf{e}_r\times\mathbf{e}_\phi)\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right]$$

In rectangular coordinates

$$\mathbf{e}_r = \begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix}, \quad \mathbf{e}_\theta = \begin{pmatrix}\cos\theta\cos\phi\\ \cos\theta\sin\phi\\ -\sin\theta\end{pmatrix}, \quad \mathbf{e}_\phi = \begin{pmatrix}-\sin\phi\\ \cos\phi\\ 0\end{pmatrix}$$

Hence for the cross products

$$\begin{aligned}\mathbf{e}_r\times\mathbf{e}_r &= 0\\ \mathbf{e}_r\times\mathbf{e}_\theta &= \mathbf{e}_\phi\\ \mathbf{e}_r\times\mathbf{e}_\phi &= -\mathbf{e}_\theta\end{aligned}$$

Angular momentum $\mathbf{L}$ now reduces to

$$\mathbf{L} = -i\hbar\left(\mathbf{e}_\phi\frac{\partial}{\partial\theta} - \mathbf{e}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)$$

Substitute for the basis vectors.

$$\mathbf{L} = -i\hbar\begin{pmatrix}-\sin\phi\\ \cos\phi\\ 0\end{pmatrix}\frac{\partial}{\partial\theta} + i\hbar\begin{pmatrix}\cos\theta\cos\phi\\ \cos\theta\sin\phi\\ -\sin\theta\end{pmatrix}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}$$

In component form

$$\begin{aligned}L_x &= i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\theta\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)\\ L_y &= i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \frac{\cos\theta\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)\\ L_z &= -i\hbar\frac{\partial}{\partial\phi}\end{aligned}$$