

Alpha decay

Let $V(r)$ be an ansatz potential for alpha decay where Z is atomic number.

$$V(r) = \begin{cases} -V_0 & 0 < r < r_1 \\ \frac{2(Z-2)\alpha\hbar c}{r} & r > r_1 \end{cases}$$

Let r_2 be a radius such that $r_2 > r_1$ and

$$V(r_2) = \frac{2(Z-2)\alpha\hbar c}{r_2} = E$$

Hence the region from r_1 to r_2 is a barrier potential for an alpha particle with energy E .

Using the WKB method we have

$$T \approx e^{-2\gamma}$$

where

$$\gamma = \frac{\sqrt{2m}}{\hbar} \int_{r_1}^{r_2} \sqrt{V(r) - E} dr$$

Factor out \sqrt{E} to obtain

$$\begin{aligned} \gamma &= \frac{\sqrt{2mE}}{\hbar} \int_{r_1}^{r_2} \sqrt{\frac{V(r)}{E} - 1} dr \\ &= \frac{\sqrt{2mE}}{\hbar} \int_{r_1}^{r_2} \sqrt{\frac{r_2}{r} - 1} dr \end{aligned}$$

Use the substitution

$$r = r_2 \sin^2 u \quad dr = 2r_2 \sin u \cos u du$$

to obtain

$$\begin{aligned} \gamma &= \frac{\sqrt{2mE}}{\hbar} 2r_2 \int_{u_1}^{u_2} \sqrt{\frac{1}{\sin^2 u} - 1} \sin u \cos u du \\ &= \frac{\sqrt{2mE}}{\hbar} 2r_2 \int_{u_1}^{u_2} \sqrt{1 - \sin^2 u} \cos u du \\ &= \frac{\sqrt{2mE}}{\hbar} 2r_2 \int_{u_1}^{u_2} \cos^2 u du \end{aligned}$$

where

$$u_1 = \arcsin \sqrt{r_1/r_2} \quad u_2 = \frac{\pi}{2}$$

Solve the integral to obtain

$$\gamma = \frac{\sqrt{2mE}}{\hbar} 2r_2 \left[\frac{u}{2} + \frac{\sin(2u)}{4} \right]_{u_1}^{u_2} \quad (1)$$

Evaluate the limits.

$$\gamma = \frac{\sqrt{2mE}}{\hbar} 2r_2 \left[\frac{\pi}{4} - \frac{1}{2} \arcsin \sqrt{r_1/r_2} - \frac{1}{4} \sin \left(2 \arcsin \sqrt{r_1/r_2} \right) \right] \quad (2)$$

From the identities

$$\sin(2x) = 2 \sin x \cos x$$

and

$$\cos(\arcsin x) = \sqrt{1 - x^2}$$

the final term simplifies as

$$\frac{1}{4} \sin \left(2 \arcsin \sqrt{r_1/r_2} \right) = \frac{1}{2} \sqrt{r_1/r_2} \sqrt{1 - r_1/r_2}$$

Hence

$$\gamma = \frac{\sqrt{2mE}}{\hbar} 2r_2 \left[\frac{\pi}{4} - \frac{1}{2} \arcsin \sqrt{r_1/r_2} - \frac{1}{2} \sqrt{r_1/r_2} \sqrt{1 - r_1/r_2} \right]$$

For ${}^{238}_{92}\text{U} \rightarrow {}^{234}_{90}\text{Th}$ we have

$$A = 238$$

$$Z = 92$$

$$E = 4.27 \text{ MeV}$$

Hence

$$\begin{aligned} r_1 &= A^{1/3} \times 1.2 \times 10^{-15} \text{ m} &&= 7.44 \times 10^{-15} \text{ m} \\ r_2 &= \frac{2(Z-2)\alpha\hbar c}{E} &&= 60.7 \times 10^{-15} \text{ m} \end{aligned}$$

Then for $m = 4.0015 \text{ u}$ we obtain

$$\gamma = 48.6$$

and

$$T = e^{-2\gamma} = 6.39 \times 10^{-43}$$

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