Alpha decay 1

Let $V(r)$ be an ansatz potential for alpha decay where $2Z$ is the product of alpha-particle charge and daughter nucleus charge.

$$V(r) = \begin{cases} -V_0 & 0 < r < r_1 \\ \dfrac{2Z\alpha\hbar c}{r} & r > r_1 \end{cases}$$

Let r_2 be a radius such that $r_2 > r_1$ and

$$V(r_2) = \frac{2Z\alpha\hbar c}{r_2} = E$$

Hence the region from r_1 to r_2 is a barrier potential for an alpha particle with energy E.

Using the WKB method we have

$$T \approx e^{-2\gamma}$$

where

$$\gamma = \frac{\sqrt{2m}}{\hbar} \int_{r_1}^{r_2} \sqrt{V(r) - E}\, dr$$

Factor out $\sqrt{E}$ to obtain

$$\begin{aligned} \gamma &= \frac{\sqrt{2mE}}{\hbar} \int_{r_1}^{r_2} \sqrt{\frac{V(r)}{E} - 1}\, dr \\ &= \frac{\sqrt{2mE}}{\hbar} \int_{r_1}^{r_2} \sqrt{\frac{r_2}{r} - 1}\, dr \end{aligned}$$

Use the substitution

$$r = r_2 \sin^2 u \qquad dr = 2r_2 \sin u \cos u$$

to obtain

$$\begin{aligned} \gamma &= \frac{\sqrt{2mE}}{\hbar} 2r_2 \int_{u_1}^{u_2} \sqrt{\frac{1}{\sin^2 u} - 1}\, \sin u \cos u\, du \\ &= \frac{\sqrt{2mE}}{\hbar} 2r_2 \int_{u_1}^{u_2} \sqrt{1 - \sin^2 u}\, \cos u\, du \\ &= \frac{\sqrt{2mE}}{\hbar} 2r_2 \int_{u_1}^{u_2} \cos^2 u\, du \end{aligned}$$

where

$$u_1 = \arcsin \sqrt{r_1/r_2} \qquad u_2 = \frac{\pi}{2}$$

Solve the integral to obtain

$$\gamma = \frac{\sqrt{2mE}}{\hbar} 2r_2 \left[\frac{u}{2} + \frac{\sin(2u)}{4} \right]_{u_1}^{u_2} \tag{1}$$

Evaluate the limits.

$$\gamma = \frac{\sqrt{2mE}}{\hbar}\, 2r_2 \left[\frac{\pi}{4} - \frac{1}{2}\arcsin\sqrt{r_1/r_2} - \frac{1}{4}\sin\left(2\arcsin\sqrt{r_1/r_2}\right)\right] \tag{2}$$

From the identities

$$\sin(2x) = 2\sin x\cos x$$

and

$$\cos(\arcsin x) = \sqrt{1-x^2}$$

the final term simplifies as

$$\frac{1}{4}\sin\left(2\arcsin\sqrt{r_1/r_2}\right) = \frac{1}{2}\sqrt{r_1/r_2}\sqrt{1-r_1/r_2}$$

Hence

$$\gamma = \frac{\sqrt{2mE}}{\hbar}\, 2r_2 \left(\frac{\pi}{4} - \frac{1}{2}\arcsin\sqrt{r_1/r_2} - \frac{1}{2}\sqrt{r_1/r_2}\sqrt{1-r_1/r_2}\right)$$

Noting that $r_2 \gg r_1$ we introduce the approximations

$$\arcsin\sqrt{r_1/r_2} \approx \sqrt{r_1/r_2}$$
$$\sqrt{r_1/r_2}\sqrt{1-r_1/r_2} \approx \sqrt{r_1/r_2}$$

and write

$$\gamma = \frac{\sqrt{2mE}}{\hbar}\, 2r_2 \left(\frac{\pi}{4} - \sqrt{r_1/r_2}\right)$$

Distribute the r_2.

$$\gamma = \frac{2\sqrt{2mE}}{\hbar}\left(\frac{\pi r_2}{4} - \sqrt{r_1 r_2}\right)$$

Substitute

$$r_2 = \frac{2Z\alpha\hbar c}{E}$$

to obtain

$$\gamma = C_1 \frac{Z}{\sqrt{E}} - C_2\sqrt{Zr_1} \tag{3}$$

where

$$C_1 = \sqrt{2m}\,\pi\alpha c \qquad C_2 = 4\sqrt{\frac{m\alpha c}{\hbar}}$$

For the alpha decay process ${}^{238}_{92}\mathrm{U} \to {}^{234}_{90}\mathrm{Th}$ we have

$$A = 238$$
$$Z = 90$$
$$E = 4.27\,\mathrm{MeV}$$

Hence

$$r_1 = A^{1/3} \times 1.2 \times 10^{-15}\,\mathrm{m} = 7.44 \times 10^{-15}\,\mathrm{m}$$

$$r_2 = \frac{2Z\alpha\hbar c}{E} = 60.7 \times 10^{-15}\,\mathrm{m}$$

Then for $m = 4.0015\,\mathrm{u}$ we obtain

$$\gamma = 47.8$$

and

$$T = e^{-2\gamma} = 3.17 \times 10^{-42}$$

Eigenmath script