Addition of angular momentum 2

In spherical coordinates

$$L_x = i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)$$
$$L_y = i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)$$
$$L_z = -i\hbar\frac{\partial}{\partial\phi}$$

and

$$L^2 = L_x^2 + L_y^2 + L_z^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

Recall that for spherical harmonic $Y_{lm}(\theta,\phi)$

$$L_z Y_{lm} = m\hbar Y_{lm}$$

Let χ_+ and χ_- be spin basis states such that

$$\chi_+ = \begin{pmatrix}1\\0\end{pmatrix}, \quad \chi_- = \begin{pmatrix}0\\1\end{pmatrix}$$

Then for $\Psi = Y_{lm}\chi_+$ we have

$$\begin{aligned} J_z\Psi &= L_z\Psi + S_z\Psi \\ &= m\hbar\Psi + \tfrac{1}{2}\hbar\Psi \\ &= \left(m + \tfrac{1}{2}\right)\hbar\Psi \end{aligned} \tag{1}$$

and for $\Psi = Y_{lm}\chi_-$

$$\begin{aligned} J_z\Psi &= L_z\Psi + S_z\Psi \\ &= m\hbar\Psi - \tfrac{1}{2}\hbar\Psi \\ &= \left(m - \tfrac{1}{2}\right)\hbar\Psi \end{aligned} \tag{2}$$