Addition of angular momentum

Let $\mathbf{J}$ be the sum of orbital angular momentum $\mathbf{L}$ and spin angular momentum $\mathbf{S}$.

$$\mathbf{J} = \mathbf{L} + \mathbf{S}$$

Recall that

$$L_x\psi = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\psi \qquad S_x\chi = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\chi$$

$$L_y\psi = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\psi \qquad S_y\chi = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\chi$$

$$L_z\psi = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\psi \qquad S_z\chi = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\chi$$

Let Ψ be the product of wave function ψ and electron spinor χ.

$$\Psi = \psi\chi$$

Then

$$\mathbf{J}\Psi = \mathbf{L}\Psi + \mathbf{S}\Psi$$

Let J^2 be the magnitude-squared of total angular momentum.

$$J^2 = \mathbf{J}\cdot\mathbf{J} = J_x^2 + J_y^2 + J_z^2$$

Operator J^2 can be decomposed as

$$J^2 = (\mathbf{L}+\mathbf{S})\cdot(\mathbf{L}+\mathbf{S}) = L^2 + S^2 + 2\mathbf{L}\cdot\mathbf{S}$$

In Eigenmath code

$$\mathbf{L}\cdot\mathbf{S}\Psi = \texttt{contract(L(S(Psi)))} \tag{1}$$

The commutation relations for J^2 are

$$[J^2, L^2] = 0$$
$$[J^2, S^2] = 0$$

$$[J^2, J_x] = 0$$
$$[J^2, J_y] = 0$$
$$[J^2, J_z] = 0$$

$$[J^2, L_x] = 2i\hbar(L_yS_z - L_zS_y)$$
$$[J^2, L_y] = 2i\hbar(L_zS_x - L_xS_z)$$
$$[J^2, L_z] = 2i\hbar(L_xS_y - L_yS_x)$$

$$[J^2, S_x] = -2i\hbar(L_yS_z - L_zS_y)$$
$$[J^2, S_y] = -2i\hbar(L_zS_x - L_xS_z)$$
$$[J^2, S_z] = -2i\hbar(L_xS_y - L_yS_x)$$